MAS114: Lecture 4

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http://cranch.staff.shef.ac.uk/mas114/

2021–2022

Another online test

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Another online test comes out at 11; there won't be many more reminders!

Functions

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I was thinking about the function from the set of people watching this lecture to the natural numbers.

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$$(g \circ f)(x) = g(f(x)).$$

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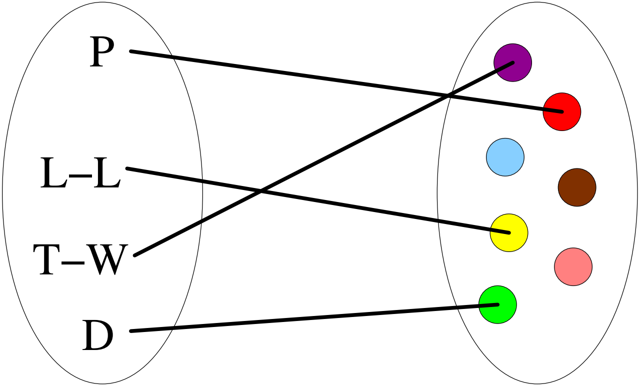
Accordingly, here is a function from the set T of Teletubbies, as considered earlier, to the set of colours:



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Here are some useful words.

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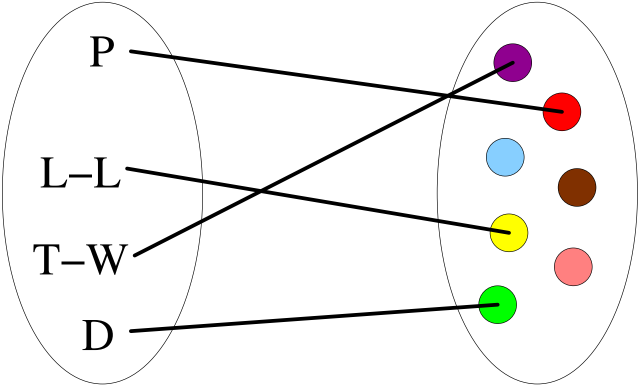

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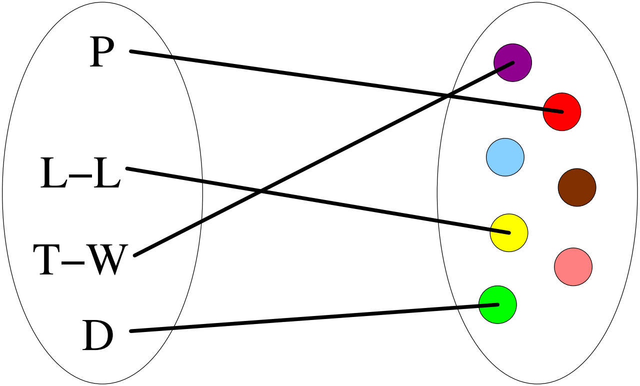

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However, it is not surjective, because there are no pink Teletubbies in all of Teletubbyland.

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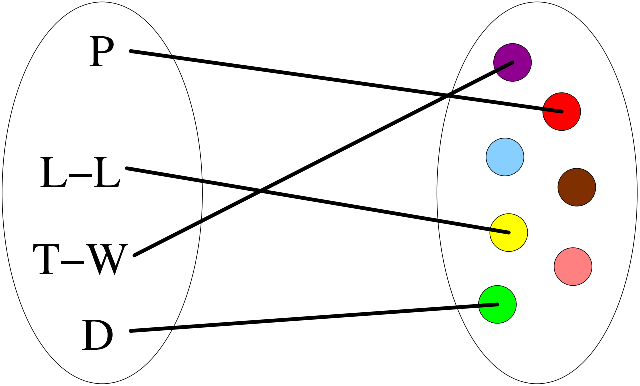

It is injective, because each one of the Teletubbies has a different colour.

However, it is not surjective, because there are no pink Teletubbies in all of Teletubbyland. Hence it is also not bijective.

Notes

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Also, note that these properties (injective, surjective, bijective) don’t just depend on the rule that defines it: they depend on the domain and codomain.

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No; $f(3) = f(-3)$.

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No; there is no $a \in \mathbb{N}$ with $g(a) = 50$.

Is it surjective as a function $g : \mathbb{Z} \to \mathbb{Z}$?

Yes it is! For any n we have $g(n - 100) = n$.

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You'll see a lot more about inverses next semester.

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However, as it happens, neither of these is true: in fact, I haven't been in Wales for a while longer than that.

As such, it is quite different to saying "A is true, and therefore B is also true". Beginning students often get these confused.

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For example, it's correct to say

> *"If $2+2=337$, then this course is lectured by Dr Cranch".*

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This may be a surprise if you're basing your intuition on ordinary English, where people use the words "if. . . then" in several different ways, sometimes slightly ambiguously.

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In fact, it's true that $2 + 2 = 4$ no matter whether it rains next Wednesday, but that's not a problem. Whether or not it's *helpful* to say that is another question!

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