MAS114: Lecture 5

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Definition

Consider a statement of the form $A \Rightarrow B$. Then the *converse* of that statement is the statement $B \Rightarrow A$.

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Yes, it is.

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Let C be the statement “Y is English”, and let D be the statement “Y lives in Sheffield”.

Is the statement $C \Rightarrow D$ always true?

No. (Y could live in New York.)

Is the converse always true?

No. (Y could be German.)

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Sometimes people shorten "if and only if" to "iff".

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Note that double negation doesn’t do anything: the statement $\neg(\neg P)$ is equivalent to P. Since statements are either true or false, if it’s not “not true”, it’s true.

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Definition

Consider a statement of the form $P \Rightarrow Q$. Its *contrapositive* is the statement $(\neg Q) \Rightarrow (\neg P)$.

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then I might rephrase it to myself as:

> *"I don't get a text at lunchtime, then Mel won't visit this evening."*

But here's a formal statement and proof, anyway.

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Proof.

I'll prove this using a truth table, showing what happens in all possibilities:

P	Q	$P \Rightarrow Q$	$\neg Q$	$\neg P$	$(\neg Q) \Rightarrow (\neg P)$
0	0				
0	1				
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You see from this that $(\neg Q) \Rightarrow (\neg P)$ is true exactly when $P \Rightarrow Q$ is, and this proves that they're equivalent.

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Let P and Q be statements. The statement $P \wedge Q$, pronounced “P and Q”, is the statement that both P and Q are true.
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While you’re all used to these words from everyday life, there can be vagueness about how “or” is used in English.
For example,

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is probably intended to mean “but not both”. In mathematical argument when we use “or” and mean “but not both”, we have to say so explicitly.

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The only way that the first statement can be false is if P is true and Q is false. But that’s also the only way that the second statement can be false, so they’re equivalent. □

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Note that that shows another style of proof of logical statements: by analysis rather than the “case bash” used in truth tables.

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$$\begin{aligned} \forall & \quad \text{for "for all";} \\ \exists & \quad \text{for "there exists";} \\ \text{s.t.} & \quad \text{for "such that".} \end{aligned}$$

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$$\exists x \in \mathbb{R} \quad \text{s.t.} \quad x^2 - 3x - 12 = 0$$

is to be read as

"There exists a real number x such that $x^2 - 3x - 12 = 0$."

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For example, the statement

$$\forall n \in \mathbb{N}, \quad \exists x \in \mathbb{R} \quad \text{s.t.} \quad x^2 = n$$

says that every natural number n has a square root x.

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This says that there's a particular number x which has the property that x is the square root of *every natural number*. And that's nonsense.