MAS114: Lecture 6

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2021–2022

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Similarly, the negation of “there exists a dolphin who likes Beethoven”is “there does not exist a dolphin who likes Beethoven”,and that’s equivalent to “all dolphins do not like Beethoven”.

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Perhaps you may want to remember that "negation swaps $\forall$ and $\exists$." But being able to *do it correctly by remembering what's going on* is much more important than remembering a slogan. After a while it should come to seem natural.

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- *true*, in which case you need to prove it in general (that's a statement with a "$\forall$" in);
- *false*, in which case you need to find a counterexample (that's a statement with a "$\exists$" in).

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> *then I can reach any rung on the ladder.*

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We call part (i) the *base case*, and part (ii) the *induction step*. These words agree quite well with our mental picture of a ladder! When we are trying to prove the induction step $P(k) \Rightarrow P(k+1)$ we refer to $P(k)$ as the *induction hypothesis*.

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We'll prove many things by induction in this course, but this is one:

Proposition

For any natural number n, we have the following formula for the sum of the first n positive integers:

$$1 + 2 + \cdots + (n-1) + n = \frac{n(n+1)}{2}.$$

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So $P(3)$ is the statement that says $1 + 2 + 3 = 3 \times 4/2$, and $P(10)$ is the statement that

$$1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 = (10 \times 11)/2.$$

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Notice that $P(n)$ is *not a number*, it's a *statement*.

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For our base case, $P(0)$ says that the sum of *no integers at all* is $0 \times 1/2$, which is true, as the sum of no integers is zero.

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$$1 + 2 + \cdots + (k-1) + k = \frac{k(k+1)}{2}.$$

We need to prove $P(k+1)$, which would say that

$$1 + 2 + \cdots + (k-1) + k + (k+1) = \frac{(k+1)(k+2)}{2}.$$

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Now note that

$$\begin{aligned}
& 1+2+\cdots+(k-1)+k+(k+1) \\
= \; & (1+2+\cdots+(k-1)+k)+(k+1) \\
= \; & \frac{k(k+1)}{2}+(k+1) \quad \text{(by the induction hypothesis)} \\
= \; & \frac{k(k+1)+2(k+1)}{2} \\
= \; & \frac{(k+1)(k+2)}{2}.
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This is exactly the statement $P(k+1)$, which is what we needed for the induction step, and that completes the proof. □

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> If you believe that induction is a reliable method of proof (and I do, and I hope you do too), then it had better be the case that we're not using induction correctly.

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$$k + 1 \qquad = (k + 83) + 1 \qquad \text{(by assumption)}$$

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$$\begin{aligned} k+1 &= (k+83)+1 && \text{(by assumption)} \\ &= (k+1)+83 && \text{(by rearrangement).} \end{aligned}$$

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If you don't have a base case, such as $P(0)$, then it's of no use to prove that $P(k) \Rightarrow P(k+1)$ for all k. It's no use to be able to climb a ladder if the bottom of the ladder is unreachable.

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We'll take $P(1)$ as the base case of the induction. This is the statement "Given any one horse, all of them have the same colour": this is obviously true.

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Now we'll prove the induction step. We will assume that $P(k)$ is true ("given any k horses, all of them have the same colour"): our job is to prove that $P(k+1)$ is true ("given any $(k+1)$ horses, all of them have the same colour").

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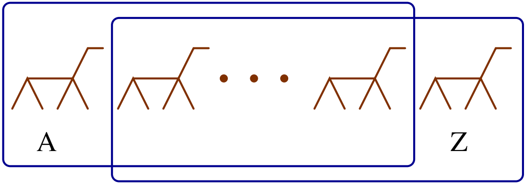

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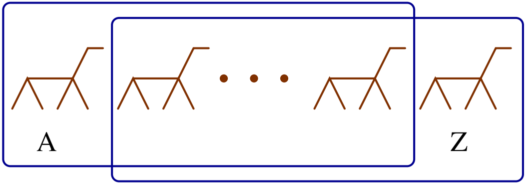

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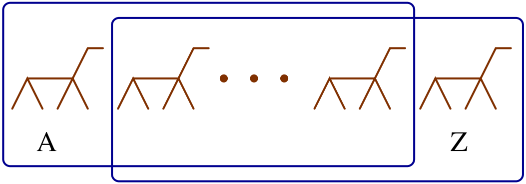

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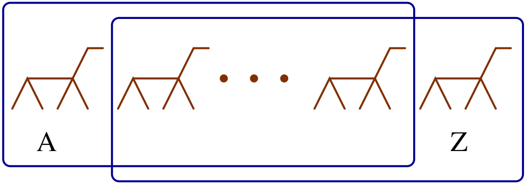

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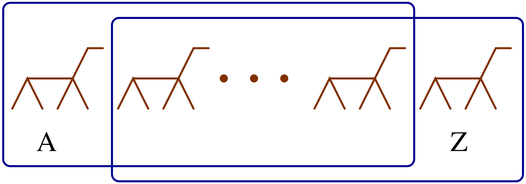

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In fact, it's a parody of a *valid* style of argument. If it is the case that *any two things are the same*, then we could prove using exactly this method that they're *all the same*. In fact, this is something you already know, since "all are alike" and "no two differ" are synonymous phrases.

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then $P(n)$ is true for all $n \geqslant 15$.
Perhaps you want to think of that as saying "if have a door which leads to the fifteenth rung of a ladder, and you know how to climb ladders, then you can get to every rung above the fifteenth".

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