MAS114: Lecture 9

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If you see a proof which you consider to be particularly well written, try to *learn from it* as an example of what's good to do. Similarly, if you see a proof which you consider to be particularly badly written, maybe you can learn something from the experience about what to avoid!
Also, *read it back* to yourself (or better yet, get a colleague to read it). You're writing it so others can read: it's good to test it to make sure this is possible. This goes particularly for proofs containing large amounts of symbols: these can be very hard to read, and reading it back to yourself is probably the best way of detecting this.

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In particular, too many people overuse the symbol "$\therefore$" (meaning "therefore"), and the symbol "$\because$" (meaning "because"). I won't use them at all in my proofs in these notes. There are so many good phrases which do its job, and choosing one helps you write what you're actually thinking. Also, it distracts the reader's eye from the symbols which matter — the actual maths you're doing — to things which don't.

Alternatives to symbols

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Here are some phrases which do the job of "$\therefore$":

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so, *hence*,

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so, *hence*, *therefore*, *thus*, *consequently*,

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so, hence, therefore, thus, consequently, as a result,

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so, hence, therefore, thus, consequently, as a result, accordingly, for this reason,

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When it's really important, I like to use both to make sure the point is clear. I might write,

> *Let* $A = \sum_{i=1}^{n} a(i)$ *be the sum of the first n values of a, and let* $p > A$ *be a prime number greater than A.*

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However, experienced readers will read it as

The square of $(5 = 25)$*.*

This is of course nonsense: equations don't really have squares, and $5 = 25$ is a false equation anyway.

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Let $P(n) = a_n = 3^n + 1$.

This has $P(n)$ as a *number*, not a *statement*. Instead, if you want to name the statement, write:

Let $P(n)$ *be the statement that* $a_n = 3^n + 1$.

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  - have a descriptive section heading.

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For example, if a proof contains an assertion that some construction is a function, then the definition of a function gives you some things to check: that *every* element of the domain gives a *unique* element lying *inside* the codomain. Unless they're obviously true, it could be that these checks are the hardest and most interesting part of the proof. They could even be lies.

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If the proof is long, then regard it as being made of several parts. Give each part an explanation when you start and when you finish it.

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It's the same with proofs: when you write them down, you're supposed to *finish* with the result you're trying to prove,

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It's quite normal to find yourself solving problems, where you start by thinking about where you want to be, and end up working out how to get there from where you are.

For example, a while ago I visited my family back home in Guernsey, and wondered about the timings of my journey from Sheffield. My plane left Gatwick at 1600, so I worked out I had to check in by 1515. Hence I wanted to be at Gatwick rail station by 1500 at the latest, so...

This was a very efficient process for working out what time to do everything (much better than trial and error, where you repeatedly guess times to leave your house, working out when you arrive). But the paragraph above is a terrible set of instructions for someone else who wants to get from Sheffield to Guernsey: it's backwards.

It's the same with proofs: when you write them down, you're supposed to *finish* with the result you're trying to prove, and on the way things are supposed to follow from your assumptions and the things you said earlier.

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- If you're trying to prove something about a sequence with a recurrence relation, your proof may well be by an induction, where the induction step refers to the same previous cases as the recurrence relation.

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Remark

That definition probably just says that a greatest common divisor is what you'd expect it to be, given the name!

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$$\gcd(123456789, 987654321)$$

by this approach!