MAS114: Lecture 19

James Cranch

http://cranch.staff.shef.ac.uk/mas114/

2021–2022

Reading group

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A second-year student is running a reading group on modern algebra: meeting in Hicks K14 at 3pm on Wednesdays. All are welcome!

An example

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Let's see an example.

Suppose Bob has low opinions of Eve's calculational skills, and chooses to use the (unrealistically small) primes $p = 101$ and $q = 103$.

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$$71 \times 431 = 30601 = 3 \times 10200 + 1.$$

Thus his private key is $pq = 10403$, $d = 431$.
Suppose Alice decides she needs to send Bob message 1245, which they've agreed in advance should mean "please meet me after this lecture".

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So she sends Bob 8763.

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Bob receives this, and his task then is to calculate 8763^{431} modulo 10403.

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So she sends Bob 8763.
Bob receives this, and his task then is to calculate 8763^{431} modulo 10403. A similar strategy makes this possible, too, and he finds that

$$8763^{431} \equiv 1245 \pmod{10403},$$

so he has reconstructed Alice's message.

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Let's set ourselves back to a time before $\mathbb{R}$ was invented, and ask: why was it necessary to invent it? Why should we feel that $\mathbb{Q}$ is not enough?
The result that set the ancient Greeks thinking was this:

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Theorem

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Proof.

We'll prove this by contradiction; suppose there is such a number $x \in \mathbb{Q}$. Because it's in $\mathbb{Q}$, it takes the form $x = p/q$ for some integers p and q with $q \neq 0$.

We may as well take p and q to be coprime ("in lowest terms").

Then we have $p^2/q^2 = x^2 = 2$, so $p^2 = 2q^2$ with p and q coprime.

Now, the right-hand side is even (it's given as a multiple of 2, so the left-hand side, p^2 must be even too. That means that p itself must be even: so we can write $p = 2r$.

Then we have $(2r)^2 = 2q^2$, which simplifies to $4r^2 = 2q^2$, or $2r^2 = q^2$. Here the left-hand side is even, so q^2 must be even. Hence q itself must be even.

This is a contradiction: p and q can't both be even. So our initial assumption is absurd, and there is no rational x with $x^2 = 2$. $\square$

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The number $\sqrt{2}$ *is not in* $\mathbb{Q}$.

But I want to flag that up as being possibly inappropriate: our aim in this section is to define the reals. We shouldn't even be confident that $\sqrt{2}$ exists yet.
However, thanks to this theorem, we can be confident at least that there's no number *inside* $\mathbb{Q}$ which deserves to be called $\sqrt{2}$.

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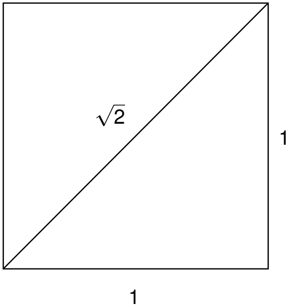
This, to the Greeks, was evidence that there was a world beyond $\mathbb{Q}$; a world of *irrational numbers* (numbers not in $\mathbb{Q}$). They needed a number called $\sqrt{2}$, so they could talk about the diagonal of a unit square:



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For the time being, and *for the time being only* we'll investigate the reals in a similar, informal way. For now, you can regard the real numbers $\mathbb{R}$ as being built out of decimals (as you did at school). In the last lecture of the course, we'll sort this out, and consider a modern construction of the reals.

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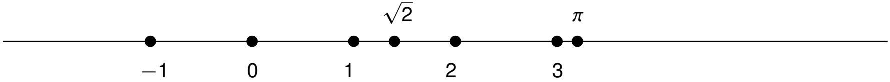

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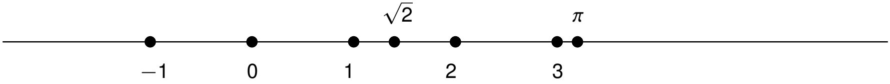

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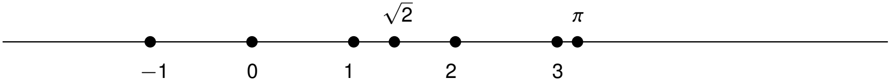

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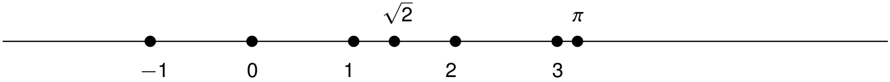

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I've also marked on $\sqrt{2}$, which we now know to be irrational, and π, which I've claimed to you is irrational: these things are in the set $\mathbb{R}\backslash\mathbb{Q}$ of irrational numbers.

In my mind, I think of the real numbers $\mathbb{R}$ as a solid line, and the rational numbers $\mathbb{Q}$ as a very fine gauze net stretched out within it.

The rationals in $\mathbb{R}$

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The reals $\mathbb{R}$ are also a lovely system of numbers, closed not just those four operations but many others: square roots (of positive numbers), sines, cosines, and so on.

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So, the irrational numbers $\mathbb{R}\backslash\mathbb{Q}$ really are just the big messy clump left over in $\mathbb{R}$ when you remove $\mathbb{Q}$.

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We prove the second one by contradiction too. Suppose that xy is rational. Then $(xy)/y$ is also rational, as it's obtained by dividing two rational numbers (with the latter nonzero), but it's equal to x which we know to be irrational. That's the contradiction we wanted. □

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It turns out that the most interesting things you can ask about are to do with *approximation*. Why is the notion of approximation so important?

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The definition will seem complicated, and probably harder to get your head around than other definitions in the course. However, that's because it really is a subtle concept: all the simpler approaches you might think of are wrong.

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A sequence $a_0, a_1, a_2, \ldots$ *converges to x* if it gets closer and closer to x. In other words, if

$$|a_0 - x| > |a_1 - x| > |a_2 - x| > \cdots .$$

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Why is this completely wrong? Well, for example, the sequence

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also gets closer and closer to 1000:

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Of course, this sequence never gets particularly close to 1000 (the sequence never goes above 4, so it never gets within 996 of 1000), but it's always getting closer!

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But this means that if our definition of “converging to x” were the completely wrong definition “gets closer and closer to x”, then the sequence

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would “converge to π”, but it would also “converge to 1000”.
But that’s not what we want: this sequence is a terrible way of getting to 1000, but a good way of getting to π.