MAS114: Lecture 20

James Cranch

http://cranch.staff.shef.ac.uk/mas114/

2021–2022

An early Christmas present

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I've put online my number theory tool, to help you revise.

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It's also linked from the main course webpage.

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Before we understand why this is wrong, we ought to make sure we know what this is supposed to mean.
I like to think of it as an argument with a very dangerous and unpleasant *evil opponent*. The evil opponent gets to choose a (positive real) distance, and we win if the sequence gets within that distance of x, and we lose if it doesn't.
In order to be *sure* of winning, we have to know how to beat the evil opponent whatever they say.

Wrong approach 2, continued

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$$3, \quad 3.1, \quad 3.14, \quad 3.141, \quad 3.1415, \quad \ldots,$$

gets to 1000, then if the evil opponent is stupid enough to ask "does the sequence get within distance 100000?" we'll win.

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It's lousy as an address for either: if we give these instructions to the numberline's village postal worker, they'll get very annoyed as they keep walking from somewhere near 1 to somewhere near 2 and back again.
But according to the definition above it would "converge" to both, because it gets as close as you like to 1 and it also gets as close as you like to 2.

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Definition

Let x be a real number. A sequence of real numbers $a_0, a_1, a_2, \ldots$ is said to *converge to x* if we have

$$\forall \epsilon > 0, \quad \exists N \in \mathbb{N} \quad \text{s.t.} \quad \forall n > N, \quad |a_n - x| < \epsilon.$$

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So that says "no matter what positive real ϵ our evil opponent gives us, we can point out some N, such that all the terms $a_{N+1}, a_{N+2}, a_{N+3}, \ldots$ are all within ϵ of x".

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That does an excellent job of making precise the concept of "gets close and stays close forever", and it's the right definition!

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Does that converge to 1000? No, it never comes within 1 of 1000 (for example), so it certainly doesn't stay within 1 of 1000 forever.

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What about the sequence

$$a_0 = 1.1, \quad a_1 = 2.01, \quad a_2 = 1.001, \quad a_3 = 2.0001, \quad \ldots?$$

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No, it doesn't. In particular, it doesn't converge to 1, because while it's sometimes close to 1, it's also sometimes close to 2. So there is no N where a_n is always within 0.1 of 1 for all $n > N$: all the odd-numbered a_n aren't in that range.

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Similarly, it doesn't converge to 2, because while it's sometimes close to 2, it's sometimes close to 1. So there is no N where a_n is always within 0.1 of 2 for all $n > N$: all the even-numbered a_n aren't in that range.

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Similarly, it doesn't converge to 2, because while it's sometimes close to 2, it's sometimes close to 1. So there is no N where a_n is always within 0.1 of 2 for all $n > N$: all the even-numbered a_n aren't in that range.
So, given the difficulties we've had in finding the right definition, perhaps you'll have some sympathy for the fact that it took about two centuries to sort real analysis out properly. In what remains of the course I'll try to make you like this definition.

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Proposition

A sequence $a_0, a_1, \ldots$ cannot converge to two different real numbers x and y.

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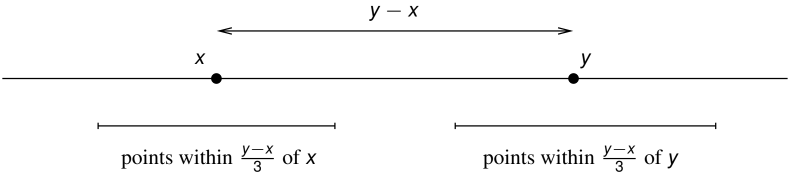

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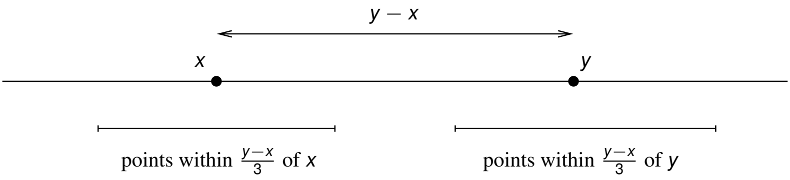

The two bars at the bottom of that diagram were deliberately chosen not to overlap (I made each of them extend one third of the distance from x to y, leaving another third of the distance between them in the middle).

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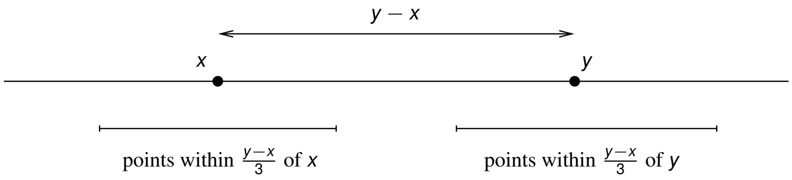

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And since it converges to x and y, eventually all the terms of the sequence should be within the interval around x, and also all of them within the interval around y, which is a contradiction since the intervals don't meet.

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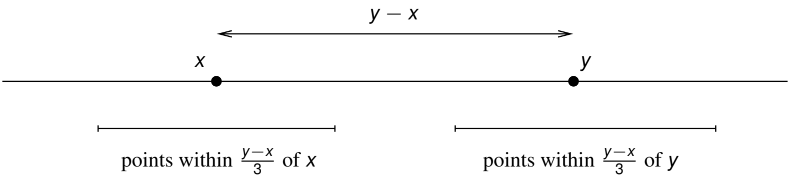

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Let's do the working carefully.

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Proof.

We'll prove this by contradiction. So, suppose it can: suppose that there is a sequence $a_0, a_1, \ldots$, which converges to two different real numbers x and y. Without loss of generality, we may take $x < y$.

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Since the sequence $a_0, a_1, \ldots$ converges to y, there is some M such that, for all $n > M$, we have $|a_n - y| < \frac{y-x}{3}$.

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Since the sequence $a_0, a_1, \ldots$ converges to y, there is some M such that, for all $n > M$, we have $|a_n - y| < \frac{y-x}{3}$.
But then, using the triangle inequality, for any n bigger than both N and M, we have

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Convergent sequences

Proof.

We'll prove this by contradiction. So, suppose it can: suppose that there is a sequence $a_0, a_1, \ldots$, which converges to two different real numbers x and y. Without loss of generality, we may take $x < y$.
Since the sequence $a_0, a_1, \ldots$ converges to x, there is some N such that, for all $n > N$, we have $|a_n - x| < \frac{y-x}{3}$.
Since the sequence $a_0, a_1, \ldots$ converges to y, there is some M such that, for all $n > M$, we have $|a_n - y| < \frac{y-x}{3}$.
But then, using the triangle inequality, for any n bigger than both N and M, we have

$$y - x = |y - x| \leqslant |y - a_n| + |x - a_n| < \frac{y-x}{3} + \frac{y-x}{3} = \frac{2}{3}(y - x)$$

which is a contradiction as $y - x$ is positive. □

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Remark

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As a result, if a sequence is convergent, there is a unique real number to which it converges; we call that the *limit* of the sequence.

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That course is about streetfighting, and there you're encouraged to use any techniques you have to hand without worrying too much about what it means.
This is a course about proof in mathematics, and tracing reasoning back to basic principles. If I set problems about convergence in MAS114, I need you to give a rigorous proof, with everything traced back to the definition of convergence (unless you're told otherwise), rather than using the slightly vaguer methods and extra theorems you saw there!