MAS114: Lecture 21

James Cranch

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2021–2022

No more online tests

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Online tests will start again next semester.

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Remember that a sequence $a_0, a_1, \ldots$ converges to x if

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We'll be continuing to work with that.

Proving convergence

Let's now try proving that some sequence or other does converge, as we're not well practiced at that yet:

A convergent sequence

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Proposition

The sequence

$$a_1 = 0, \quad a_2 = 1/2, \quad a_3 = 2/3, \quad a_4 = 3/4, \quad a_5 = 4/5, \quad \ldots$$

where $a_n = \frac{n-1}{n}$, *converges to* 1.

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Proof (rough version).

The definition of convergence is complicated, so it may be helpful to start by reminding us what we're aiming for. So we'll start by working from the wrong end.

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We need to show that, for every $\epsilon > 0$, there is some N, such that for all $n > N$ we have

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A natural thing to do is to simplify that:

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So, as we can see, what we're aiming for is that, for every $\epsilon > 0$ there is some N, such that for all $n > N$ we have

$$\frac{1}{n} < \epsilon.$$

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So if we take N to be $\lceil \frac{1}{\epsilon} \rceil$, the smallest integer greater than $1/\epsilon$, that works. □

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exactly as required. □

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That second version is obviously correct, and all the reasoning goes in the right direction. But analysis proofs often have the property that the most persuasive proof will seem a bit mysterious. It’s best to do the rough work and then rewrite it neatly.

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One useful one is the following:

Theorem (Sandwich Lemma)

Suppose we have three sequences: $a_0, a_1, a_2, \ldots$, *and* $b_0, b_1, b_2, \ldots$ *and* $c_0, c_1, c_2, \ldots$, *such that:*

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(i) *the sequences* $(a_i)_{i \in \mathbb{N}}$ *and* $(c_i)_{i \in \mathbb{N}}$ *both converge to the same number* x*; and*

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(i) *the sequences $(a_i)_{i\in\mathbb{N}}$ and $(c_i)_{i\in\mathbb{N}}$ both converge to the same number x; and*

(ii) *for all i we have*

$$a_i \leqslant b_i \leqslant c_i$$

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Then the sequence $(b_i)_{i\in\mathbb{N}}$ also converges to x.

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Given that, I claim we can take N to be $\max(M, P)$, the larger of M and P. Then we have

$$b_n - \epsilon \leqslant c_n - \epsilon < x < a_n + \epsilon \leqslant b_n + \epsilon,$$

as required. Here the outer two inequalities are because $a_i \leqslant b_i \leqslant c_i$ for all i, and the inner two are obtained from the convergence of $(a_i)_{i\in\mathbb{N}}$ and $(c_i)_{i\in\mathbb{N}}$. □

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However, we can proceed by sandwiching it between $a_n = 3 - \frac{1}{n}$ and $c_n = 3 + \frac{1}{n}$ (since all values of sin are always between -1 and 1). Showing that those two sequences both converge to 3 is no harder than the examples we've done already.

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Theorem

Let $a_0, a_1, \ldots$ be a sequence that converges to x, and let $b_0, b_1, \ldots$ be a sequence that converges to y. Then the sequence

$$a_0 + b_0, \quad a_1 + b_1, \quad \ldots$$

converges to $x + y$.

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If a sequence ought to converge to something, then its terms ought to "settle down" somehow. That means they should at least get close at least to each other.

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In vaguer terms, a sequence is Cauchy if no matter what we mean by close, there is some point beyond which all the terms of the sequence are close to each other.