MAS114: Lecture 22

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Of course, it's very unusual to be able to choose one N that works for every ϵ.

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Suppose we have a sequence $a_0, a_1, \ldots$ converging to x. We must show that it is Cauchy.

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Since it is convergent, there is an N such that for all $n > N$ we have

$$|a_n - x| < \frac{\epsilon}{2}.$$

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$$\begin{aligned} |a_m - a_n| &\leqslant |a_m - x| + |a_n - x| \qquad \text{(by the triangle inequality)} \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon, \end{aligned}$$

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Let's give it in context. What follows is *revisionist history*: things didn't actually happen exactly like this, but maybe they should have done.

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  We've said only a little about *constructing* the naturals, but we could have said more.

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  But that's okay: the theory is workable, and we get $\mathbb{Z}$ by doing it. Nobody gets confused about which integers are equal. We can make definitions like

$$a - b = c - d \qquad \text{if and only if } a + d = b + c.$$

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  Again, we don't actually want one new number for each Cauchy sequence. There are other Cauchy sequences of rationals that converge to π (some of them more interesting, perhaps). A famous example is due to Gregory and Leibniz:

$$4, \quad 4-\frac{4}{3}, \quad 4-\frac{4}{3}+\frac{4}{5}, \quad 4-\frac{4}{3}+\frac{4}{5}-\frac{4}{7}, \quad \ldots.$$

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But then, given that, we could just say that the reals *are* the Cauchy sequences of rationals, subject to some restriction about which ones are the same.

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We might regard this as saying “no matter what is meant by close, the two sequences get close to each other and stay close to each other forever”.

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In particular, I hope you agree that it's a more natural way of understanding the reals than talking about decimal expansions.

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(This happens to be the decimal expansion of the square root of 2.) Then this can be accommodated in our construction easily, using the trick I mentioned earlier: we can represent it as the limit of the Cauchy sequence

$$1, \quad \frac{14}{10}, \quad \frac{141}{100}, \quad \frac{1414}{1000}, \quad \frac{14142}{10000}, \quad \ldots .$$

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If we start with 1 as an approximation, then this gives us the sequence

$$1, \quad \frac{3}{2}, \quad \frac{19}{12}, \quad \frac{577}{408}, \quad \frac{665857}{470832}, \ldots$$

It's not hard to imagine that this is a much better way of describing $\sqrt{2}$ than its decimal expansion: easier to prove things about it than some weird string of digits.

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So far as number theory goes, our elementary methods will give out sooner or later. A good next step is to learn lots of algebra. (There are other good reasons to do that.) Next semester, Evgeny will take this up. You can return later in your degrees to a huge range of questions of which equations are solvable in which systems of numbers.

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Sooner or later, you can try using the same techniques in more exotic surroundings: the concepts of approximation we started talking about in this course give us a way in to studying abstract concepts of space.